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CALCULUS: DIFFERENTIATION

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CALCULUS

Calculus is the branch of mathematics focused on the calculation of instantaneous rates of change (**differentiation** or differential calculus) and the summation of infinitely many small factors to determine some whole (**integration** or integral calculus). The creation of calculus is widely acknowledged to have occurred independently and simultaneously through the work of two distinguished figures in the seventeenth century: Sir Isaac Newton, renowned for his contributions to gravity, and the philosopher-mathematician Gottfried Leibniz.

DIFFERENTIATION

Differentiation is the process of finding the derivative, or rate of change, of a function. For example, with differentiation, you can find the rate of change of velocity with respect to time (i.e., acceleration). It particularly allows us to find the rate of change of y with respect to x, which on a graph of y against x is the gradient of the curve.

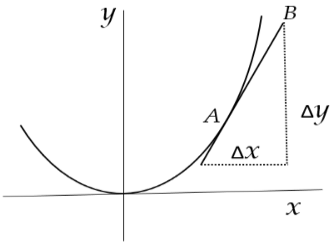
DERIVATION FROM FIRST PRINCIPLES

Suppose we want to find the gradient of a curve at a point A in Figure 1.1. We can add another point B on the line close to point A. Let Δx represents a small change in x and Δy represents a small change in y. As point B moves closer to point A, the gradient of the line AB gets closer to the gradient of the tangent at A.

We can write the gradient of the line AB as:

$$\frac{\textbf{change in } \boldsymbol{y}}{\textbf{change in } \boldsymbol{x}} = \frac{\boldsymbol{\Delta y}}{\boldsymbol{\Delta x}} = \frac{\boldsymbol{\delta y}}{\boldsymbol{\delta x}}$$

Figure 1.1 Gradient of a curve at a point A



As B gets closer to A, δx gets closer to 0 and $\frac{\delta y}{\delta x}$ gets closer to the value of the gradient of the tangent at A. However, δx cannot be equal to 0 because we would then have division by 0 and the gradient would be undefined. Rather we consider the limit as δx tends to 0. This means that δx becomes infinitesimal without actually becoming 0.
In general,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$\frac{dy}{dx}$ represents the derivative of y with respect to x.

According to the general principles, the derivative of a function $y = f(x)$ at a point x is

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \lim_{\delta x \to 0} \frac{\boldsymbol{f(x + \delta x) - f(x)}}{\boldsymbol{\delta x}}$$

provided that the limit exists.

NOTE: At a given point on a curve, the **gradient of the curve** is equal to the gradient of the tangent to the curve. The derivative describes the gradient of a curve at any point on the curve. It also describes the gradient of a tangent to the curve at any point.

☑ EXAMPLE 1.1

Find the derivative of the following functions using first principles.

a) $y = x^2$ b) $y = x^3 - 7x + 1$

SOLUTION **tips**

a) $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^2 - x^2}{\delta x} = \lim_{\delta x \to 0} \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x}$$