10

CALCULUS: DEFINITE INTEGRALS

CONTENTS

The **definite integral** is defined as $\int_a^b f(x)dx$. The numbers a and b are known as the lower and upper limits of the integral. Definite integrals have many applications, for example in finding areas bounded by curves and consumer surplus.

THE FUNDAMENTAL THEOREM OF CALCULUS

The *fundamental theorem of calculus,* expressed mathematically, states that

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where $[F(x)]_a^b$ indicates that b and a are to be substituted successively for x.

Properties of Definite Integrals

1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2) $\int_a^a f(x)dx = F(a) - F(a) = 0$
3) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
4) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

☑ EXAMPLE 10.1

Evaluate $\int_{-1}^{2} 5x\,dx$.

SOLUTION tips

First perform the integration and use this convention.

$$\int_{-1}^{2} 5x\,dx = \left[\frac{5x^2}{2}+c\right]_{-1}^{2}$$

Let x equal the value at the upper limit, then the value at the lower limit.

$$\int_{-1}^{2} 5x\,dx = \left[\frac{5(2)^2}{2}+c\right]-\left[\frac{5(-1)^2}{2}+c\right] = 10-\frac{5}{2} = 7\frac{1}{2}$$

☑ EXAMPLE 10.2

Find

a) $\int_{1}^{4}\left(3\sqrt{x}+\frac{4}{\sqrt{x}}\right)\,dx$ b) $\int_{4}^{8}(4t-1)^{-\frac{1}{3}}\,dt$ c) $\int_{-1}^{1} xe^{4x^2-3}dx$

SOLUTION tips

a) $$\int_{1}^{4}\left(3\sqrt{x}+\frac{4}{\sqrt{x}}\right)\,dx = \int_{1}^{4}\left(3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)\,dx$$
$$= \left[\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{4x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 = \left[2x^{\frac{3}{2}}+8x^{\frac{1}{2}}\right]_1^4$$
$$= (2\cdot 8+8\cdot 2)-(2\cdot 1+8\cdot 1) = 22$$

b) $$\int_{4}^{8}(4t-1)^{-\frac{1}{3}}\,dt = \frac{1}{4}\cdot\frac{3}{2}\left[(4t-1)^{\frac{2}{3}}\right]_4^8 = \frac{3}{8}\left[(4(8)-1)^{\frac{2}{3}}\right]-\frac{3}{8}\left[(4(4)-1)^{\frac{2}{3}}\right] = 1.42$$

c) Use integration by substitution

Let $u = 4x^2-3$ then, $\frac{du}{dx} = 8x$ $dx = \frac{1}{8x}du$

It follows that

$$\int_{-1}^{1} xe^{4x^2-3}dx = \int_{-1}^{1} xe^{u}\left(\frac{1}{8x}du\right) = \frac{1}{8}\int_{-1}^{1} e^{u}du = \left[\frac{1}{8}\,e^{u}\right]_0^1$$
$$= \left[\frac{1}{8}e^{4x^2-3}\right]_0^1 = \left[\frac{1}{8}e^{4(1)^2-3}\right]-\left[\frac{1}{8}e^{4(0)^2-3}\right]$$
$$= \frac{e-e^{-3}}{8} = 0.33$$

☑ EXAMPLE 10.3

Find the integrals

a) $\int_{0}^{1}\frac{x^2+7x-14}{x^2+2x-15}dx$ b) $\int_{1}^{3}\frac{4x}{(2x-1)^2(x+1)}dx$