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Quadratic Forms, Definite Matrices, Eigenvalues & Hessian

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Quadratic Forms and Definite matrix

The quadratic form of an $n \times n$ symmetric matrix A is given by

$$Q(x) = x^T A x$$

where x is a vector of variables, and A is a symmetric matrix.

For example, consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

The quadratic form is given by

$$Q(x) = x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1^2 + 4x_1x_2 + x_2^2$$

Also consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The quadratic form is given by

$$Q(x) = x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 2x_2^2 + 3x_3^2$$

The quadratic form $Q(x)$ is:

- Positive definite if $Q(x) > 0$ for all $x \neq 0$.
- Negative definite if $Q(x) < 0$ for all $x \neq 0$.
- Indefinite if $Q(x)$ assumes both positive and negative values.

Also, $Q(x)$ is said to be positive definite if $Q(x) \geq 0$ for all x, and negative definite if $Q(x) \leq 0$ for all x.

The matrix A will determine to which one of the above definitions the quadratic form will belong. We can use either of the two tests:

1. Discriminant test
2. Eigenvalues test

For example, matrix A is positive definite if and only if all the eigenvalues are positive (see Table 13.1). The determinant of a matrix is the product of its eigenvalues. So, if all the eigenvalues are positive, then the discriminant/determinant is also positive.

Table 13.1 Conditions for Definiteness

Definiteness	Discriminant/Determinant of Principal Minors	Eigenvalues
positive definite	$D_i > 0$, $i = 1 \ldots n$	all $r_i > 0$
positive semidefinite	$D_i \geq 0$, $i = 1 \ldots n$	all $r_i \geq 0$
negative definite	$D_1 < 0$, $D_2 > 0$, $D_3 < 0 \ldots$	all $r_i < 0$
negative semidefinite	$D_1 \leq 0$, $D_2 \geq 0$, $D_3 \leq 0 \ldots$	all $r_i \leq 0$
indefinite	none of the above	some $r_i \geq 0$, some $r_i \leq 0$

☑ **EXAMPLE 13.2**

Write $Q = 5x_1^2 + 4x_2^2 - 4x_1x_2$ in the form $Q(x) = x^TAx$, where A is a 2×2 matrix.

SOLUTION**tips**

The cross term is $-4x_1x_2 = -2x_1x_2 - 2x_2x_1$. Hence

$$Q(x) = x^TAx = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

NOTE: The coefficients of the squares appear on the main diagonal while the coefficients of the cross terms are neatly divided among two positions to give a symmetric matrix. Hence this matrix is referred to as the matrix of the quadratic form Q.