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DUALITY IN LINEAR PROGRAMMING

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DUALITY: PRIMAL & DUAL

The duality in linear programming indicates that every maximization problem has a corresponding minimization problem and vice versa. The original linear programming problem is called “**Primal**,” while the derived linear problem is called “**Dual**.” If the primal objective function is maximized, then its dual is minimized.

The von Neumann Duality Principle, after the American mathematician John von Neumann, defines the primal-dual relation.

> **The Von Neumann duality principle** or **the strong duality theorem:** The optimal solution of a primal linear programming problem, if it exists, has the same value as the optimal solution of the dual.
>
> $$\mathbf{max}_x \boldsymbol{c^T x} = \mathbf{min}_y \boldsymbol{c^T y}$$

The optimal solutions for the primal and dual problems are equivalent. Maximising profits in the primal by finding the optimal outputs (x_1, x_2) is comparable to minimizing the total shadow prices or opportunity cost of the resource inputs (y_1, y_2) in the dual.

For example, given the primal maximization problem:

$$\text{Max } Z = 5x_1 + 6x_2$$

$$\text{subject to } \begin{cases} x_1 + 3x_2 \leq 7 \\ 2x_1 + 4x_2 \leq 8 \end{cases} \quad x_1, x_2 \geq 0$$

The dual minimization problem is:

Min $C = 7y_1 + 8y_2$

subject to $\begin{cases} y_1 + 2y_2 \geq 5 \\ 3y_1 + 4y_2 \geq 6 \end{cases}$ $\quad y_1, y_2 \geq 0$

NOTE: The coefficient matrix of the constraints of the primal is the transpose of the coefficient matrix of the constraints of the dual. If the objective of the primal is to be maximized, the objective of the dual is to be minimized. The elements of the right-hand side of the constraints in the primal are the respective coefficients of the objective function in the dual.

Duality is a powerful tool for analyzing and solving optimization problems, and has a wide range of applications in various fields, such as economics, engineering, and operations research.

Dual LP problems can be solved using different methods. Here, we discuss the matrices and simplex methods.

MATRIX METHOD

If the number of decision variables is equal to the number of constraints, Cramer's rule is easily applied to a dual LP problem. Cramer's rule is a method for solving systems of linear equations using determinants.

☑ EXAMPLE 18.1

Jon would like to optimize her food expenses. On sale at Amazon Supermarket are brownies for $8/ounce and ice cream for $16/ounce. She wants to ensure she gets enough cheese and chocolate in her diet, for which the recommended monthly amounts are 52 ounces and 12 ounces respectively. The table gives the amounts of these requirements in each food item.

	Brownies	Ice Cream
Cheese	7	3
Chocolate	1	5

a) Determine the least expensive combination of brownies and ice cream that meet his requirements. Also, what is the least cost?

b) Assuming Blockbuster Delicacies supplies the supermarket with the chocolate and cheese needed to make the treats. In order to maximize revenue, how can the supplier set the prices so that the supermarket still buys from him? Also, what is the maximum revenue?

SOLUTION **tips**

a) Let y_1 = the ounces of brownies bought

y_2 = the ounces of ice cream bought

The primal problem is

Min $C = 8y_1 + 16y_2$ subject to $\begin{cases} 7y_1 + 3y_2 \geq 52 \\ y_1 + 5y_2 \geq 12 \end{cases}$ $\quad y_1, y_2 \geq 0$

Set up the minimization problem in matrix form

$$\begin{bmatrix} 7 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 52 \\ 12 \end{bmatrix}$$

Use Cramer's rule. Find the determinant of matrix:

$$|A| = \begin{vmatrix} 7 & 3 \\ 1 & 5 \end{vmatrix} = 32 \quad |A_{y_1}| = \begin{vmatrix} 52 & 3 \\ 12 & 5 \end{vmatrix} = 224 \quad |A_{y_2}| = \begin{vmatrix} 7 & 52 \\ 1 & 12 \end{vmatrix} = 32$$