20

NON-LINEAR PROGRAMMING AND ENVELOPE THEOREM

CONTENTS

Nonlinear programming is the optimization of a nonlinear objective function subject to constraints, which can include linear constraints, or nonlinear constraints. Non-linear programming extends the techniques of constrained optimization by allowing inequality constraints into the problem, especially constraints that may not be binding in the solution.

CONCAVE PROGRAMMING

Concave programming refers to a type of optimization problem where the objective function is a concave function and the feasible region is a convex set. Concave programming, a form of non-linear programming, is used to optimize functions subject to inequality constraints. The objective and constraint functions are assumed concave. Since the negative of a convex function is concave, it also considers convex functions. Concave programming can minimize a function by maximizing the negative of that function. The goal is to find the point in the feasible region that minimizes the objective function. Some applications of concave programming include portfolio optimization, resource allocation, and revenue management.

KARUSH-KUHN-TUCKER (KKT) CONDITIONS

The **Karush-Kuhn-Tucker** (KKT) conditions, also known as the **Kuhn-Tucker** (KKT) conditions, are a set of first-order necessary conditions for solving non-linear optimization problems (with equality and inequality constraints), including concave programming problems.

The Kuhn–Tucker approach to nonlinear programming extends the Lagrange multiplier method by accommodating inequality constraints in addition to

equality constraints. Similar to the Lagrange approach, the optimization problem with constraints is reformulated as a Lagrange function. The optimal point of this Lagrange function represents a global saddle point, serving as both a global maximum (minimum) over the domain of choice variables and a global minimum (maximum) over the multipliers. This is why the Karush–Kuhn–Tucker theorem is also known as the saddle-point theorem. Originally introduced by Harold W. Kuhn and Albert W. Tucker in 1951, the Kuhn–Tucker conditions were later found to have been articulated by William Karush in his 1939 master's thesis.

Given an optimization problem:

maximize $f(x_1, x_2)$

Subject to $g(x_1, x_2) \leq 0$ $\quad x_1, x_2 \geq 0$

The Lagrangian function is,

$$Z = f(x_1, x_2, \lambda) + \lambda g(x_1, x_2)$$

The *Kuhn-Tucker* conditions are

1. a) $\frac{\partial Z}{\partial x_i} = f_i(\bar{x}_1, \bar{x}_2) + \bar{\lambda} g_i(\bar{x}_1, \bar{x}_2) \leq 0$ ($\geq$ for a minimization problem)

   b) $x_i \geq 0$

   c) $\bar{x}_i \frac{\partial Z}{\partial x_i} = 0 \qquad i = 1,2$

2. a) $\frac{\partial Z}{\partial \lambda} = g(\bar{x}_1, \bar{x}_2) \geq 0$ ($\leq$ for a minimization problem)

   b) $\bar{\lambda} \geq 0$

   c. $\bar{\lambda} \frac{\partial Z}{\partial \lambda} = 0$

where the condition $\bar{\lambda} \frac{\partial Z}{\partial \lambda} = 0$ is called the *complementary-slackness condition*, meaning that both $\bar{x}$ and $f'(\bar{x})$ cannot simultaneously both be nonzero.

The rationale for the conditions (a) to (c) is demonstrated in the three scenarios Figure 20.1 For a value x to give a local maximum, it must satisfy the following three conditions:

A: interior solution	$f'(x) = 0$	and	$x > 0$
C: boundary solution	$f'(x) = 0$	and	$x = 0$
D or F: both boundary solutions	$f'(x) < 0$	and	$x = 0$

The three conditions can be summarized as

$$f'(x) = 0 \qquad x \geq 0 \qquad \text{and} \qquad xf'(x) = 0$$

which are obviously part of the Kuhn-Tucker conditions.