21

DIFFERENTIAL EQUATIONS

CONTENTS

GENERAL FORMULA

A **differential equation** is a mathematical equation that relates a function with its derivatives.
Suppose we are given

$$\frac{dy}{dt} = 2y$$

It is a **first-order linear differential equation**: first-order because it involves only the first derivative (dy/dt); linear because neither y nor its derivative is raised to any power other than 1.

The general form of a first-order linear differential equation is

$$\frac{dy}{dt} + ay = b$$

where a and b may be constants or functions of time.

The formula for a general solution is

$$\boldsymbol{y(t) = e^{-\int a dt}\left(A + \int b e^{\int a dt} dt\right)}$$

where A is an arbitrary constant.
The solution consists of two components on the right-hand side: a **complementary function** ($e^{-\int adt}A$) and a **particular integral** ($e^{-\int adt}\int be^{\int adt}$). The particular integral y_p represents the intertemporal

equilibrium level of y, and the complementary function y_c, the deviations of the time path from that equilibrium. For $y(t)$ to be dynamically stable, y_c must approach zero as t approaches infinity.

It is important to note that the solution of the differential equation is a function of t (a time path: a corresponding value of y can be calculated by substituting for a particular value of t), and the solution $y(t)$ contains no derivatives.

NOTE: The solution of a differential equation can always be checked by differentiation.

☑ EXAMPLE 21.1

Solve the equation $\frac{dy}{dt} + 2y = 6$, with the initial condition $y(0) = 4$.

SOLUTION**tips**

Compare $\frac{dy}{dt} + 2y = 6$ to $\frac{dy}{dt} + ay = b$.
Since $a = 2$ and $b = 6$

$$y(t) = e^{-\int 2dt}\left(A + \int 6e^{\int 2dt}dt\right)$$

$\int 2dt = 2t + c$. Since c is absorbed by A,

$$y(t) = e^{-2t}(A + \int 6e^{2t}dt)$$

Integrate the remaining integral: $\int 6e^{2t}dt = 3e^{2t} + c$. Ignore the constant again,

$$y(t) = e^{-2t}(A + 3e^{2t}) = Ae^{-2t} + 3 \qquad \text{Since } e^{-2t}e^{2t} = e^0 = 1.$$

As $t \to \infty, y_c = Ae^{-2t} \to 0$ and $y(t)$ approaches $y_p = 3$, the intertemporal equilibrium level. Thus, $y(t)$ is dynamically stable. Note that this is a general solution considering that A is not specified. The definite solution is calculated as follows:

$$\text{At } t = 0, y(0) = 4 \text{ such that}$$

$$4 = Ae^{-2(0)} + 3 \quad \to \quad 4 = \mathrm{A} + 3 \qquad \text{Since } \mathrm{e}^0 = 1$$

$$A = 1$$

Substitute $\mathrm{A} = 1$ in $y(t) = Ae^{-2t} + 3$, the definite solution is $y = e^{-2t} + 3$.

☑ EXAMPLE 21.2

Solve the equation $\frac{dy}{dt} = 5$.

SOLUTION**tips**

Compare $\frac{dy}{dt} = 5$ to $\frac{dy}{dt} + ay = b$. Here $a = 0$ and $b = 5$. Thus,

$$y(t) = e^{-\int 0dt}\left(A + \int 5e^{\int 0dt}dt\right)$$

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