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DIFFERENCE EQUATIONS

CONTENTS

A **difference equation** is an equation that shows the relationship between successive values of a sequence and the differences among them. It defines the pattern of change of y between consecutive periods. Similar to differential equations, difference equations can be either linear or nonlinear, and of the first or second (or higher) orders.

Suppose we are given

$$y_{t+1} - y_t = 3$$

It is a first-order linear difference equation: first-order because it contains only a first difference involving a one-period time lag only; linear because neither y nor its lags is raised to any power other than 1.

While differential equations is applied in the continuous-time context where the pattern of change of a variable y is described by its derivative, difference equations is applied in the discrete-time context where the variable t takes only integer values. In difference equations, the pattern of change of the variable y changes only when the variable *t* changes from one integer value to the next, such as from $t = 1$ to $t = 2$. The pattern of change is represented by the difference quotient $\frac{\Delta y}{\Delta t}$ or (Δy_t), which is the discrete-time equivalent of the derivative $\frac{dy}{dt}$.

The first difference is thus:

$$\Delta y_t = y_{t+1} - y_t$$

Where y_t represents the value of y in the *t*th period, and y_{t+1} its value in the period immediately after.

An example of differential equations:

$$\Delta y_t = 3$$

Which can be written as

$$y_{t+1} - y_t = 3$$

Or $$y_{t+1} = y_t + 3$$

ITERATIVE METHOD

Analogous to solving a differential equation, the objective is to find a time path. Iteration of the pattern of change specified in the difference equation will permit us to infer a time path.

☑ EXAMPLE 22.1

Find the solution of the difference equation $y_{t+1} - y_t = 5$, assuming an initial value of $y_0 = 2$.

SOLUTION**tips**

Rearrange

$$y_{t+1} = y_t + 5$$

By iteration:

$$y_1 = y_0 + 5$$
$$y_2 = y_1 + 5 = (y_0 + 5) + 5$$
$$= y_0 + 2(5)$$
$$y_3 = y_2 + 5 = [y_0 + 2(5)] + 5$$
$$= y_0 + 3(5)$$

Generalize for any period t

$$y_t = y_0 + t(5) = 2 + 5t$$

☑ EXAMPLE 22.2

Solve the difference equation $y_{t+1} = 0.7y_t$, assuming an initial value of $y_0 = 3$.

SOLUTION**tips**

By iteration

$$y_1 = 0.7y_0$$
$$y_2 = 0.7y_1 = 0.7(0.7y_0) = (0.7)^2 y_0$$
$$y_3 = 0.7y_2 = 0.7(0.7)^2 y_0 = (0.7)^3 y_0$$

Generalize for any period t

$$y_t = (0.7)^t y_0 = 3(0.7)^t$$

GENERAL METHOD

Given the first-order difference equation

$$y_{t+1} + ay_t = c$$

Where a and c are constants.

The general solution consists of two components: a particular integral and a complementary function. The particular integral represents the intertemporal equilibrium level of y, and the complementary function, the deviations of the time path from that equilibrium.

	Case $a \neq -1$	Case $a = -1$
General Solution	$y_t = A(-a)^t + \frac{c}{1+a}$	$y_t = A + ct$
Definite Solution	$y_t = \left(y_0 - \frac{c}{1+a}\right)(-a)^t + \frac{c}{1+a}$	$y_t = y_0 + ct$

"Math practice makes math perfection."

Olaniyi Evans