23

DIFFERENTIAL & DIFFERENCE EQUATIONS OF HIGHER ORDER

CONTENTS

Recall that the order of a differential/difference equation is the highest derivative/lag in the equation. After studying first-order equations in previous chapters, it is natural to look at higher-order equations. We therefore embark on the analysis of second-order equations because they arise in applications more often than do other higher-order equations. Observe that all the results from second-order equations can be extended to other higher-order (e.g., third-order, fourth-order, etc) equations.

SECOND-ORDER DIFFERENTIAL EQUATIONS

A second-order linear differential equation has the form

$$y''(t) + a_1 y'(t) + a_2 y(t) = b$$

where a_1, a_2, and b are constants.

To solve a second-order differential equations, calculate the complementary function y_c and the particular integral y_p separately. Together, the general solution is: $y(t) = y_c + y_p$
The particular integral is

$$y_p = \frac{b}{a_2} \qquad a_2 \neq 0$$

$$y_p = \frac{b}{a_1} t \qquad a_2 = 0 \qquad a_1 \neq 0$$

$$y_p = \frac{b}{2}t^2 \qquad a_1 = a_2 = 0$$

The complementary function is

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

And

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Where A_1 and A_2 are arbitrary constants; r_1 and r_2 are the *characteristic roots*.

☑ EXAMPLE 23.1

Given $y''(t) - 3y'(t) + 2y(t) = 10$, (a) find the general solution and (b) the definite solution when $y(0) = 15$ and $y'(0) = 14$.

SOLUTION **tips**

a) Compare $y''(t) - 3y'(t) + 2y(t) = 10$ to the general form of a second-order linear differential equation $y''(t) + a_1 y'(t) + a_2 y(t) = b$.

Thus, the particular integral is

$$y_p = \frac{b}{a_2} = \frac{10}{2} = 5$$

The complementary function is calculated thus,

$$r_1, r_2 = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)}}{2} = \frac{3 \pm 1}{2} = 1, 2$$

Substitute in $y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$

$$y_c = A_1 e^{t} + A_2 e^{2t}$$

The general solution consists of two components: the complementary function y_c and the particular integral y_p, that is, $y(t) = y_c + y_p$.

$$y(t) = A_1 e^{t} + A_2 e^{2t} + 5$$

b) To find the definite solution

Differentiate $y(t) = A_1 e^{t} + A_2 e^{2t} + 5$

$$y'(t) = A_1 e^{t} + 2A_2 e^{2t}$$

At $t = 0$, $y(0) = 15$ and $y'(0) = 14$,

$$y(0) = A_1 e^{0} + A_2 e^{2(0)} + 5 = 15 \qquad \text{thus} \qquad A_1 + A_2 = 10$$

$$y'(0) = A_1 e^{0} + 2A_2 e^{2(0)} = 14 \qquad \text{thus} \qquad A_1 + 2A_2 = 14$$

Solve simultaneously: $A_1 = 6$ and $A_2 = 4$.