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SIMULTANEOUS SYSTEMS OF DIFFERENTIAL & DIFFERENCE EQUATIONS

CONTENTS

In this chapter, we want to take a look at systems of differential/difference equations that are 2×2. These equations can be solved by writing them in matrix form, which are then solved as standard differential/difference equations.

SIMULTANEOUS DIFFERENTIAL EQUATIONS: CASE I

Given the following system of first-order linear differential equations.

$$y_1' = a_{11}y_1 + a_{12}y_2 + b_1$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + b_2$$

The system is called homogeneous if all b_i = 0, otherwise it is called non-homogeneous.
In matrix form

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$Y' = AY + B$$

The general solution of the system will compose of 2 equations, each with (i) a complementary solution y_c and (ii) a particular solution y_p.

The complementary solution takes the form,

$$\boldsymbol{y_c = k_1 V_1 e^{r_1 t} + k_2 V_2 e^{r_2 t}}$$

where k_i is a scalar or constant, $\boldsymbol{V}_i$ is (2 x 1) column vector of constants (also known as the *eigenvector*), and r_i is a scalar (also known as the *characteristic root*).

The characteristic roots (also known as *eigenvalues)* are computed by solving the quadratic equation

$$\boldsymbol{r_i} = \frac{\boldsymbol{Tr(A)} \pm \sqrt{[\boldsymbol{Tr(A)}]^2 - 4|\boldsymbol{A}|}}{\mathbf{2}}$$

where $|A|$ is the determinant of **A**, Tr(**A**) is the *trace* of **A** and also the sum of all the diagonal elements of **A**.
For **A**,

$$\boldsymbol{Tr(A)} = \boldsymbol{a_{11}} + \boldsymbol{a_{22}}$$

The solution of the system of simultaneous equations requires the *eigenvalue problem:*

$$(\boldsymbol{A} - \boldsymbol{r_i I})\boldsymbol{V_i} = \mathbf{0}$$

where $(A - r_i I) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - r_i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} - r_i & a_{12} \\ a_{21} & a_{22} - r_i \end{bmatrix} = 0$

The particular integral, y_p, is simply the intertemporal or steady-state solution.

$$\boldsymbol{y_p} = -\boldsymbol{A}^{-1}\boldsymbol{B}$$

where A^{-1} is the inverse of A. B is the column of constants.

The stability of the model is determined by the characteristic roots:

$r_i < 0$: dynamically stable
$r_i > 0$: dynamically unstable

In the case where r_i has different signs, the solution is said to be at a saddle-point equilibrium and the model is deemed unstable, except along the saddle path.

☑ EXAMPLE 24.1

Solve the following system of first-order, autonomous, linear differential equations,

$$y_1' = 2y_1 - 3y_2 - 10 \qquad y_1(0) = 18$$

$$y_2' = -y_1 + 4y_2 - 15 \qquad y_2(0) = 11$$

SOLUTION**tips**

In matrix form

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -10 \\ -15 \end{bmatrix}$$

$$Y' = AY + B$$

The general solution of the system will compose of *2* equations, each with (i) a complementary solution y_c and (ii) a particular solution y_p.

The complementary solution takes the form,

$$y_c = k_1 V_1 e^{r_1 t} + k_2 V_2 e^{r_2 t}$$

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