25

OPTIMAL CONTROL THEORY

CONTENTS

Optimal control theory, an extension of the calculus of variations, is a branch of mathematics that deals with finding the best way to control a dynamic system over time such that an objective function is optimized. It is useful for solving dynamic optimization problems expressed in continuous time. The system being controlled is typically described by a set of differential equations, and the control policy is a function that specifies the inputs to the system over time.

For example, the dynamical system might be a nation's economy, with the objective to minimize inflation; the controls in this case could be interest rate and money supply. For example, the dynamical system might be a firm's flow of profits over time, with the objective to maximize the total profits; the controls in this case could be the output and price.

Largely based on the work of Lev Pontryagin and Richard Bellman in the 1950s, the optimal control theory has numerous applications in science, engineering and operations research. It is used to design controllers for systems such as aircraft, spacecraft, and robots, and to optimize resource allocation in industries such as energy and transportation.

A control problem includes a functional that is a function of state and control variables. An optimal control is a set of differential equations that describe the paths of the control variables that minimize the functional. The optimal control is derivable using Pontryagin's maximum principle (a necessary condition), or by solving the Hamilton-Jacobi-Bellman equation (a sufficient condition). In optimal control theory, the objective is to find the optimal time path for the **control variable**, y. The **state variable**, x, helps us to predict the behavior of the control variable and has **equations of motion (*or* transition)** set equal to x'.

Optimal control theory problems could involve continuous time, a finite time horizon, and fixed endpoints. A standard optimal control problem is of the form:

Maximize $U = \int_0^T f[x(t), y(t), t]\, dt$
subject to $x' = g[x(t), y(t), t]$ (1)
$x(0) = x_0 \quad x(T) = x_T$

where U *is* the functional to be optimized; $y(t)$ the control variable selected or controlled to optimize U; $x(t)$ the state variable which varies over time consistent with the differential equation set equal to x' in the constraint; and t time.

THE HAMILTONIAN & THE PONTRYAGIN'S MAXIMUM PRINCIPLE

The **Hamiltonian function**, similar to the Lagrangian function, is a technique that combines the functional (being optimized) with the state variable (describing the constraint or constraints) into a single equation.
In line with (1), the Hamiltonian is

$$H[x(t), y(t), \lambda(t), t] = f[x(t), y(t), t] + \lambda(t) g[x(t), y(t), t] \quad (2)$$

where H denotes the Hamiltonian, $\lambda(t)$ is the *costate* variable (similar to the Lagrangian multiplier) which represents the marginal value or shadow price of the state variable $x(t)$.

Pontryagin's Maximum Principle

For maximization of the Hamiltonian in (2), the necessary conditions are derived from the maximum principle:

i. $\frac{\partial H}{\partial y} = 0$
ii. $\frac{\partial x}{\partial t} = x' = \frac{\partial H}{\partial \lambda}$ state equation
iii. $\frac{\partial \lambda}{\partial t} = \lambda' = -\frac{\partial H}{\partial x}$ costate equation
iv. $x(0) = 0 \qquad x(T) = x_T$ boundary condition
v. $\lambda(T) = 0$ transversality condition

The first three conditions are known as the *maximum principle*. The two equations of motion in conditions (*ii)* and (*iii)* are referred to as the *Hamiltonian system* or the *canonical system*. Condition (v) is applicable to a free endpoint problem only.
For minimization of the Hamiltonian, multiply the objective function by -1.

The sufficient conditions

The sufficient conditions are fulfilled if:

1. The objective functional and the constraint are continuously differentiable and jointly concave in x and y.
2. $\lambda(t) \geq 0$, when the constraint is non-linear in x or y. If the constraint is linear, λ can assume any sign.

   A linear function is both concave and convex, but neither strictly concave nor strictly convex. For a nonlinear function, the discriminant can test for joint concavity.

   Given the discriminant of the second-order derivatives of a function,

$$|D| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Purchase the full book at:
https://unimath.5profz.com/

We donate 0.5% of the book sales every year to charity, forever. When you buy University **Mathematics (I & II)** *you are helping orphans and the less privileged.*