4

MULTIVARIATE FUNCTIONS AND PARTIAL DERIVATIVES

CONTENTS

Multivariable calculus (also called multivariate calculus) deals with functions of several variables. In the previous chapters, differentiation was limited to functions of one independent variable only. More realistic economic models are, however, usually functions of more than one variable.

Functions of two variables are written in general form as

$$z = f(x, y)$$

For example,

$$z = x - 3y + 1 \qquad\qquad z = x^2 + y^3 - 4xy - 5$$

Where x and y are the independent variables and z is the dependent variable.

RULES OF PARTIAL DIFFERENTIATION

All rules of ordinary differentiation apply. To show that something has been partially (instead of fully) differentiated we use the symbol ∂. At a time, we differentiate only one independent variable and keep all other independent variables constant. Thus, a **partial derivative** of a function of several variables is its derivative with respect to one variable, with the others held constant.

Power rule: Given $z = x^n y^m$

$\frac{\partial z}{\partial x} = [y^m]\frac{\partial}{\partial x}(x^n) = nx^{n-1}y^m$ (Consider y as a constant)

$\frac{\partial z}{\partial y} = [x^n]\frac{\partial}{\partial y}(y^m) = my^{m-1}x^n$ (Consider x as a constant)

Generalized Power Function rule: Given $z = [g(x,y)]^n$ where g is a differentiable function:

$\frac{\partial z}{\partial x} = n[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial x}$ (Consider y as a constant)

$\frac{\partial z}{\partial y} = n[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial y}$ (Consider x as a constant)

Product rule: Given $z = g(x,y) \cdot h(x,y)$ where g and h are differentiable functions:

$\frac{\partial z}{\partial x} = g(x,y) \cdot \frac{\partial h}{\partial x} + h(x,y) \cdot \frac{\partial g}{\partial x}$ (Consider y as a constant)

$\frac{\partial z}{\partial y} = g(x,y) \cdot \frac{\partial h}{\partial y} + h(x,y) \cdot \frac{\partial g}{\partial y}$ (Consider x as a constant)

Quotient rule: Given $z = \frac{g(x,y)}{h(x,y)}$ where g and h are differentiable functions and $h \neq 0$:

$\frac{\partial z}{\partial x} = \frac{h(x,y) \cdot \frac{\partial g}{\partial x} - g(x,y) \cdot \frac{\partial h}{\partial x}}{[h(x,y)]^2}$ (Consider y as a constant)

$\frac{\partial z}{\partial y} = \frac{h(x,y) \cdot \frac{\partial g}{\partial y} - g(x,y) \cdot \frac{\partial h}{\partial y}}{[h(x,y)]^2}$ (Consider x as a constant)

☑ EXAMPLE 4.1

Find the partial derivatives of the following functions:

a) $z = 5x^2 - xy + y^4$ b) $z = -w^2 + wxy - x^3y^2$

SOLUTION**tips**

a) Apply the power rule:

$\frac{\partial z}{\partial x} = 10x - y + 0$ y is a constant

$= 10x - y$

$\frac{\partial z}{\partial y} = 0 - x + 4y^3$ x is a constant

$= -x + 4y^3$

b) $\frac{\partial z}{\partial x} = 0 + wy - 3x^2y^2$ w and y are constants

$= wy - 3x^2y^2$

$\frac{\partial z}{\partial y} = 0 + wx - 2x^3y$ w and y are constants

$= wx - 2x^3y$

$\frac{\partial z}{\partial w} = -2w + xy - 0$ x and y are constants

$= -2w + xy$

☑ EXAMPLE 4.2

Consider the functions:

a) $z = (5x - y)^3$ b) $z = \left(x^3 - \frac{3}{2}xy^2\right)^2$

Differentiate z partially with respect to x and y.

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