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COMPLEX NUMBERS

CONTENTS

Complex numbers are numbers that include a real part and an imaginary part, and they can be represented in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit, satisfying the equation $i^2 = -1$.
If we have $x^2 + 1 = 0$, there is no solution for x in real numbers. The set of complex numbers caters for such, by defining $i = \sqrt{-1}$. Complex variables make algebra simple, and the use of complex variables is an indispensable tool in the modelling of financial markets.

ARITHMETIC OPERATIONS WITH COMPLEX NUMBERS

Addition: Add real parts; add imaginary parts.

$$z_3 = z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$
$$= (a_1 + a_2) + (b_1 + b_2)i$$

Subtraction: Subtract real parts; subtract imaginary parts

$$z_4 = z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i)$$
$$= (a_1 - a_2) + (b_1 - b_2)i$$

Multiplication: Use the distributive property and remember that $i^2 = -1$.

$$z_5 = z_1 \times z_2 = (a_1 + b_1 i) \times (a_2 + b_2 i)$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$$

Division: Multiply both the numerator and denominator by the complex conjugate of the denominator, then simplify

$$z_6 = \frac{z_1}{z_2} = \frac{z_1\, \bar{z}_2}{z_2\, \bar{z}_2}$$

It is like the method for rationalising a surd.
Note here that if $z_1 = a + bi$, and $z_2 = a - bi$ then z_2 is said to be the complex conjugate of z_1. If $z_1 = -3 + 4i$ and $z_2 = -3 - 4i$, then z_2 is said to be the complex conjugate of z_1.

Square Root

$$(a \pm bi)^2 = (a^2 - b^2) \pm 2ab(i)$$

Therefore,

$$\boldsymbol{a \pm bi = \pm\sqrt{(a^2 - b^2) \pm 2ab(i)}}$$

Absolute Value or Modulus: The modulus of $z_1 = |z_1|$.
If $z_1 = a + bi$, then

$$\boldsymbol{|z_1| = \sqrt{a^2 + b^2}}$$

☑ EXAMPLE 32.1

If $z_1 = 2 - 3i$ and $z_2 = -5 - 4i$, evaluate a) $z_1 + z_2$ b) $z_1 - z_2$

SOLUTION tips

a) $z_1 + z_2 = (2 + (-5)) + (-3 + (-4)i = -3 - 7i$
b) $z_1 - z_2 = (2 - (-5)) + (-3 - (-4)i = 7 + i$

☑ EXAMPLE 32.2

Evaluate

a) $(1 + 3i)(2 - i)$ b) $\left(\frac{1}{2} + \frac{1}{4}i\right)\left(\frac{3}{2} + 8i\right)$

SOLUTION tips

a) $(1 + 3i)(2 - i) = 2 - i + 6i - 3i^2 = 2 + 5i - 3(-1) = 5 + 5i$ $i^2 = -1$
b) $\left(\frac{1}{2} + \frac{1}{4}i\right)\left(\frac{3}{2} + 8i\right) = \frac{3}{4} + 4i + \frac{3}{8}i + 2i^2 = \frac{3}{4} + \frac{35}{8}i + 2(-1) = -\frac{5}{4} + \frac{35}{8}i$

☑ EXAMPLE 32.3

Find $z_1 \div z_2$ when $z_1 = 4 + 3i$ and $z_2 = 1 - i$.

SOLUTION tips

$$z_1 \div z_2 = \frac{4 + 3i}{1 - i}$$

The complex conjugate of z_1 is $\bar{z}_2 = 1 + i$. Multiply both the numerator and denominator by the complex conjugate of the denominator.

$$z_1 \div z_2 = \frac{4 + 3i}{1 - i} \times \frac{1 + i}{1 + i} = \frac{4 + 4i + 3i + 3i^2}{1 - i^2} \qquad i^2 = -1$$
$$= \frac{1 + 7i}{2} = \frac{1}{2} + \frac{7}{2}i$$

☑ EXAMPLE 32.4

Find $z_1 \div z_2$ when $z_1 = 1 + 6i$ and $z_2 = 4 + i$.

SOLUTION tips

$$z_1 \div z_2 = \frac{(1 + 6i)(4 - i)}{(4 + i)(4 - i)} = \frac{4 - i + 24i - 6i^2}{16 - i^2}$$
$$= \frac{10 + 23i}{17} = \frac{10}{17} + \frac{23}{17}i$$

☑ EXAMPLE 32.5

If $z_1 = 3 + 2i$. Find $|z_1|$.

SOLUTION tips

$$|z_1| = \sqrt{a^2 + b^2} = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$$

☑ EXAMPLE 32.6

Find the square root of 8 – $6i$.

SOLUTION tips

Let $z^2 = (a - bi)^2 = 8 - 6i$
$(a^2 - b^2) + 2ab(i) = 8 - 6i$
Compare real parts and imaginary parts,
$a^2 - b^2 = 8$ (1)
$2ab = -6$ (2)
Solve (1) and (2) simultaneously: