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MATHEMATICAL INDUCTION

CONTENTS

Mathematical Proof Technique

Mathematical induction is a mathematical proof technique used to prove that a property is true for every natural numbers $\mathbb{N}$, i.e. for n = 0, 1, 2, 3, ... In other words, mathematical induction proves statements true for all natural numbers by starting with a base case and showing that if true for any case, it's true for the next. Metaphorically, mathematical induction is often illustrated by reference to the sequential effect of falling dominoes (The first domino falls. When any domino falls, the next domino falls. So... all dominos will fall!).

Principle of Mathematical Induction

Proof by induction consists of two cases:

- *Base Case:* Proves the statement for $n = 0$ or sometimes $n = 1$, establishing truth for specific cases.
- *Induction Step:* If statement holds for $n = k$, then it must hold for $n = k + 1$, ensuring truth for subsequent cases.

These steps establish the statement's truth for all natural numbers.

NOTE: Mathematical induction rigorously proves a general statement by a finite chain of deductive reasoning involving n, unlike inductive reasoning, which draws probable conclusions from examining many cases.

☑ EXAMPLE 30.1

Prove (for all integers $n \geq 1$) that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION**tips**

Base case:

For $n = 1$, the statement reduces to $1^2 = \frac{1(2)(3)}{6} = 1$ and is obviously true.

Induction Step:

Assuming the statement is true for $n = k$:

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad (1)$$

the statement must be true for $n = k + 1$:

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \qquad (2)$$

The LHS of (2) can be written as

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

LHS of (2) = RHS of (2). Therefore, by the principle of mathematical induction, the given statement is true for all positive integers $n \geq 1$.
CHECK: For $n = 4$,

$$1^2 + 2^2 + 3^2 + 4^2 = 30 = \frac{4(5)(9)}{6}$$

☑ Example 30.2

Prove that $2n + 1 < 2^n$ for all integers $n > 3$.

SOLUTION**tips**

Base case:
For $n = 4$, the statement reduces to $2(4) + 1 = 9 < 2^4 = 16$ and is obviously true.
Induction Step:
Assume the statement is true for $n = k$: $2k + 1 < 2^k$ (1)
The statement must be true for $n = k + 1$: $2(k+1) + 1 < 2^{k+1}$ (2)
By the induction hypothesis, $2k < 2^k$, and also $3 < 2^k$ if $k > 3$.

$$2(k+1) + 1 = 2k + 2 + 1 = 2k + 3 < 2^k + 2^k = 2^{k+1}$$

Thus, LHS of (2) is less than the RHS of (2). By the principle of mathematical induction, the given statement is true for all $n > 3$.

☑ Example 30.3

Prove that $2^{2n} - 1$ is divisible by 3 for all integers $n \geq 1$.

SOLUTION**tips**

Base case:
For $n = 1$, the statement reduces to $2^{2(1)} - 1 = 4 - 1 = 3$ and is obviously true.
Induction Step:
Assuming the statement is true for $n = k$: $2^{2k} - 1$ is divisible by 3
The statement must be true for $n = k + 1$: $2^{2(k+1)} - 1$ is divisible by 3
But the expression can transform as follows:

$$2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k}.2^2 - 1$$
$$= 2^{2k}.2^2 - 1 = 2^{2k}(3+1) - 1 = 2^{2k}.3 + (2^{2k} - 1)$$

The last expression must be divisible by 3 since, by the inductive hypothesis, $2^{2k} - 1$ is divisible by 3, and obviously, $2^{2k}.3$ is divisible by 3. Therefore, by the principle of mathematical induction, the given statement is true for all integers $n \geq 1$.

☑ Example 30.4

Prove that

$$\sum_{i=0}^{n} r^i = 1 + r + r^2 + r^3 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}; r \neq 1$$

for all integers $n \geq 0$.

SOLUTION**tips**

Base case: For $n = 0$, the statement reduces to $\sum_{i=0}^{0} r^i = r^0 = \frac{r^{0+1}-1}{r-1} = 1$ and is obviously true.
Induction Step: Assuming the statement is true for $n = k$:

$$\sum_{i=0}^{k} r^i = 1 + r + r^2 + r^3 + \cdots + r^k = \frac{r^{k+1} - 1}{r - 1} \quad (1)$$

is true for all positive integers $k \geq 1$.
The statement must be true for $n = k + 1$:

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