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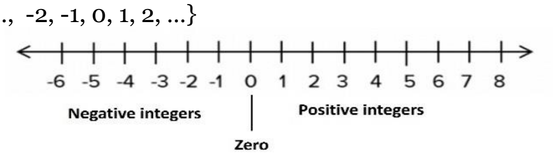
MATHEMATICAL PRELIMINARIES

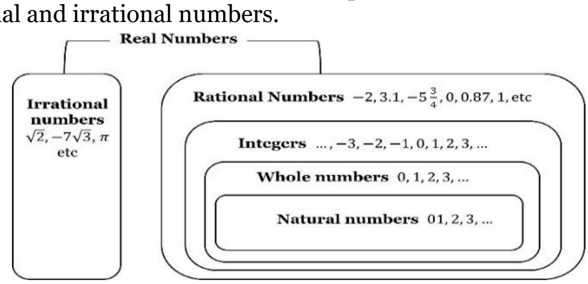
CONTENTS

NUMBER SYSTEMS

Natural numbers: {1, 2, 3, 4, ...}
Whole numbers: {0, 1, 2, 3, ...}
Integers: {..., -2, -1, 0, 1, 2, ...}



Rational numbers: all numbers that can be written in the form a/b, where a and b are integers and $b \neq 0$.
Irrational numbers: numbers that cannot be written as the quotient of two integers but can be represented on the number line.
Real numbers: all numbers that can be represented on the number line, that is, all rational and irrational numbers.



Prime numbers: a number greater than 1 that has only itself and 1 as factors.
Examples: 2, 3, and 7 are prime numbers.

Composite numbers: A natural number or a positive integer which has more than two factors.
Example: 15 has factors 1, 3, 5 and 15, hence it is a composite number.

KEY PROPERTIES

Properties of Addition

Identity Property of Zero: $a + 0 = a$
Inverse Property: $a + (-a) = 0$
Commutative Property: $a + b = b + a$
Associative Property: $a + (b + c) = (a + b) + c$

Properties of Multiplication

Property of Zero: $a \times 0 = 0$
Identity Property of One: $a \times 1 = a, when\ a \neq 0.$
Inverse Property: $a \times \frac{1}{a} = 1, when\ a \neq 0.$
Commutative Property: $a \times b = b \times a$
Associative Property: $a \times (b \times c) = (a \times b) \times c$

Properties of Division

Property of Zero: $\frac{0}{a} = 0, when\ a \neq 0.$
Property of One: $\frac{a}{a} = 1, when\ a \neq 0.$
Identity Property of One: $\frac{a}{1} = a \times 1$

ABSOLUTE VALUE

The absolute value of a number is always ≥ 0.

If $a > 0, |a| = a$. If $a < 0, |-a| = a$.

Example: $|-2| = 2$ and $|2| = 2$. In each case, the answer is positive.

BASIC WORDS/SYMBOLS

The following words and symbols are used for these operations.

Addition
Sum, plus, total, increase
addend + addend = sum

Subtraction
Difference, minus, decrease
minuend -- subtrahend = difference

Multiplication
Product, times, of,
factor × factor = product

Division
Quotient, divided by, per,
dividend ÷ divisor = quotient

INTEGERS

Adding and Subtracting with Negatives

$$-a - b = (-a) + (-b)$$
$$-a + b = b - a$$
$$a - (-b) = a + b$$

Examples:

$$-4 - 6 = (-4) + (-6) = -10$$
$$-11 + 7 = 7 - 11 = -4$$

MULTIPLYING/DIVIDING WITH NEGATIVES

$$-a \times b = -ab$$
$$-a \div b = -\frac{a}{b}$$
$$-a \times -b = ab$$
$$\frac{-a}{-b} = \frac{a}{b}$$

Examples:

$$-1 \times 2 = -2$$
$$-8 \div 4 = -\frac{2}{1} = -2$$
$$-4 \times -3 = 12$$
$$\frac{-9}{-3} = \frac{3}{1} = 3$$

ORDER OF OPERATIONS: PEMDAS

Follow this order to evaluate mathematical expressions accurately!

1st: ***Parentheses***

Simplify any expressions inside parentheses or brackets.

2nd: ***E**xponents*

Work out any exponents or powers.

3rd: ***M**ultiplication and **D**ivision*

Solve all multiplication and division, working from left to right.

4th: ***A**ddition and **S**ubtraction*

Solve all multiplication and division, working from left to right.

For example,

$$4 - 1 \times 2 + (14 - 5) \div 3^2 = 4 - 1 \times 2 + 9 \div 9 = 4 - 2 + 1 = 3$$

FRACTIONS

Fractions are used to express division. The top number of a fraction is the **numerator**, and the bottom number is the **denominator**.

Least Common Multiple (LCM): The smallest number that is a multiple of all the specified numbers.

Example: The LCM of 4 and 5 is 20, since 4 and 5 have no factors in common.

Greatest Common Factor (GCF): The largest number that can be equally divided into each of the given numbers.

Example: The GCF of 15 and 20 is 5, since both 15 and 20 are divisible by 3, however they are not both divisible by any numbers greater than 5.

Adding and Subtracting Fractions

Before fractions can be added or subtracted, they must have the same denominator.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \text{ if } c \neq 0 \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}, \text{ if } c \neq 0$$

If the fractions have different denominators, first convert into like fractions by finding the same LCM denominator of these fractions and convert them into equivalent fractions.

Example:

$$\frac{1}{2} + \frac{4}{3} = \frac{3}{6} + \frac{8}{6} = \frac{11}{6}$$

The unlike fractions are converted into equivalent fractions e*quivalent fractions* are found by multiplying the numerator and denominator of the fraction by the same number. In the example above:

$$\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$$

Multiplying and Dividing Fractions

To multiply fractions, take the product of the numerators and the product of the denominators:

$$\frac{a}{c} \times \frac{b}{d} = \frac{a \times b}{c \times d} = \frac{ab}{cd}$$

To divide fractions, invert the second fraction and then multiply the numerators and denominators:

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \times \frac{d}{b} = \frac{ad}{bc}$$

Examples:

$$\frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$$

$$\frac{3}{2} \div \frac{3}{5} = \frac{3}{2} \times \frac{5}{3} = \frac{15}{6} = \frac{5}{2}$$

Reducing Fractions

To **reduce** a fraction, divide the numerator and denominator by common factors. In the last *example* above*:*

$$\frac{15}{6} = \frac{15 \div 3}{6 \div 3} = \frac{5}{2}$$

.

Mixed Numbers

A mixed number has two parts: an integer and a fraction. An example of a mixed number is $2\frac{1}{3}$, which is same as

$$2 + \frac{1}{3} = \frac{6}{3} + \frac{1}{3} = \frac{7}{3}$$

Or equally as

$$\frac{10}{5}+\frac{1}{3}=\frac{7}{3}$$

Likewise, improper fractions can be expressed as a mixed number.

Example:

$\frac{11}{4}$ can be written as $2\,\frac{3}{4}$

Which means 11 divided by 4 equals 2 with a remainder of 3.

PROPORTIONS, PERCENTS, RATES, AND RATIOS

Rates and Ratios

A **rate** is the ratio between two related quantities in different units. For example, a jet that travels 2700 kilometers in 3 hours is moving at a rate of 2700 kilometers/3 hours or 900 km/h.

A **ratio** is a comparison of two quantities (with the *same* units). For example, a class with 2 boys and 3 girls has a boy–girl ratio of 2: 3 or $\frac{2}{3}$.

Proportions

A proportion says two ratios or rates are equal. A proportion is read as "x is to y as z is to w".

$$\frac{x}{y}=\frac{z}{w} \text{ where } y, w \neq 0$$

If one number in a proportion is unknown, you can easily find that number by solving the proportion.

Example:

To make 10 pancakes requires 2 eggs. How many eggs are required to make 500 pancakes?

We can solve this problem by taking a cue from this equation:

$$\frac{x}{500}=\frac{2}{10} \qquad \text{Cross multiply} \qquad x=\frac{2\times 500}{10}=100$$

So, 100 eggs are required to make 500 pancakes.

Percents

A percent is the number of parts in every hundred. To write a percent as a fraction, divide by 100 and remove the percent sign.

Example:

$$25\%=\frac{25}{100}=\frac{1}{4}$$

To write a fraction as a percent, multiply the fraction by 100 and attach a percent sign.

Example:

$$\frac{3}{5}=\frac{3}{5}\times 100\%=60\%$$

To find a percent of a quantity, multiply the quantity by the percent.

Example: 25% of 40

$$=\frac{25}{100}\times 40=\frac{1000}{100}=10$$

Percents to Decimals and Decimals to Percents

To change a number from a percent to a decimal, divide by 100 and remove the percent sign:

$$75\% \;=\; 75/100 \;=\; 0.75$$

To change a number from a decimal to a percent, multiply the number by 100 and attach the percent sign:

$$0.32 \;=\; 0.32\times 100 \;=\; 32\%$$

SCIENTIFIC NOTATION

Scientific notation is a math expression used to represent a decimal number between 1 and 10 multiplied by ten to the appropriate power, so as to write large numbers using less digits.
Example: 5000 written as 5×10^3
To change a number to scientific notation, move the decimal point to the right of the first digit. If the decimal point is moved n places to the left, n is positive; otherwise, n is negative. If the decimal point is not moved, n is 0.
Example: $0.000902 = 9.02 \times 10^{-4}$

Multiplying/Dividing in Scientific Notation

To multiply or divide numbers in scientific notation, multiply or divide first the decimal parts and then the powers of 10.
Example:

$$\begin{aligned}(4.0 \times 10^5) \times (1.9 \times 10^{-4}) &= (4.0 \times 1.9) \times (10^5 \times 10^{-4}) \\ &= 7.6 \times 10^{5+(-4)} = 7.6 \times 10^1\end{aligned}$$

ALGEBRAIC TERMS

Variable: a quantity that can change. Letters are used to represent these changing, unknown quantities. For example, the temperature at different times of the day is changing, so it could be represented by a variable, x.
Constant: a term that does not change. *Example:* the number of days in a week, 7, is unchanging, so it is a constant.
Expression: a group of numbers, variables, and operators (such as + and ×) representing a quantity or an operation. *Example:* $9x - \frac{2}{5}$ is an expression.
Equation: a mathematical statement that two expressions are equal.
Example: $4 - 11x = 0$ is an equation.
Solution: Any and all value(s) of the variable(s) that satisfies an equation.
Example: In $\frac{4}{x} - 1 = \frac{1}{x}$, we know that the statement is true if $x = 3$.

METRIC CONVERSIONS

12 inches = 1 feet	3 feet = 1 yard
1760 yards = 1 mile	5280 feet = 1 mile
2 cups = 1 pint	1 cups = 8 ounces
4 quarts = 1 gallon	2 pints = 1 quart
2000 pounds = 1 ton	16 ounces = 1 pound

Metric

1000 millimeters = 1 meter	100 centimeters = 1 meter
1000 meters = 1 kilometer	100 centiliters = 1 liter
1000 milliliter = 1 liter	100 centigram = 1 gram
1000 milligrams = 1 gram	1000 g = 1 kilogram
0.001 m = 1 millimeter	0.01 m = 1 centimeter
0.001 g = 1 milligram	0.01 g = 1 centigram
0.001 liter = 1 millimeter	0.01 liter = 1 centiliter

Metric/Imperial Units

8 kilometers = 5 miles	1 meter = 40 inches
30 centimeters = 1 foot	2.5 centimeters = 1 inch
1 kilogram = 2.2 pounds	1 liter = $1\frac{3}{4}$ pints
1 gallon = $4\frac{1}{2}$ liters	1 acre = $1\frac{2}{5}$ hectare
450 gram = 1 pound	

APPROXIMATIONS

There are three main ways to round numbers:

- To the nearest 10, 100, 1000, etc
- To a certain number of significant figures

- To a certain number of decimal places

Example:

$23547 = 24500$ to the nearest 100. $23547 = 24000$ to the nearest 1000.	$84259 = 84300$ to 3 significant figures $0.006197 = 0.00619$ to 3 significant figures
$0.11538 = 0.12$ to 2 decimal places. $290.183 = 290.18$ to 2 decimal places	By convention, we normally round 0.5 up to the next whole number.

PERCENTAGE ERROR & DEGREE OF ACCURACY

Percent error is the difference between an estimated value and the true or exact value, expressed in percentage.

$$\text{Percentage error} = \frac{|\text{Approximate value} - \text{Exact value}|}{|\text{Exact value}|} \times 100\%$$

Example:

If a student measured the temperature of boiling water to be 98.5°C, the percentage error is:

$$\text{Percentage error} = \frac{|98.5 - 100|}{|100|} \times 100\% = \frac{1.5}{100} \times 100\% = 1.5\%$$

The degree of accuracy is a measure of how exact a stated value is to the actual value being described. Accuracy may be affected by rounding, the use of significant figures or ranges in measurement.

Absolute error is the difference between the actual and the measured value (symbol $\pm$).

$$\text{Relative error} = \frac{|\text{absolute error}|}{\text{measured value}}$$

$$\text{Percentage error} = \frac{\text{absolute error}}{\text{measured value}} \times 100\%$$

Example:

A truck is measured as 22.5 meters long, accurate to 0.1 of a meter.

Accurate to 0.1 m means it could be up to $\frac{0.1}{2} = 0.05$ m either way:

Length = 22.5±0.05 m

So it could really be anywhere between 22.45 m and 22.55 m long.

Thus

$$\text{Absolute error} = 0.05$$

$$\text{Relative error} = \frac{0.05}{22.5} \cong 0.002$$

$$\text{Percentage error} = \frac{0.05}{22.5} \times 100\% \cong 0.2\%$$

MODULAR ARITHMETIC

Modular arithmetic is a method of arithmetic for integers, where numbers "loop" when reaching a certain value, called the **modulus**.

A common use of modular arithmetic is in the 12-hour clock, in which the day is divided into two 12-hour periods. If the time is 6:00 now, then 7 hours later it will be 1:00. In normal addition, the later time should be 6 + 7 = 13, but the hour number starts over when it reaches 12; this is arithmetic modulo 12.

For a positive integer n, two numbers a and b are congruent modulo n, if their difference $a - b$ is an integer multiple of n. This congruence relation is denoted: $a \equiv b \pmod n$ where n is the modulus of the congruence.

More precisely, the statement $a \equiv b \pmod n$ implies that a and b have the same remainder when divided by n.

Example:

$37 \equiv 13 \pmod{12}$

Because $37 - 13 = 24$, which is a multiple of 12. Equally, because both 37 and 13 have the same remainder 1 when divided by 12.

The same rule applies for negative values:

$-9 \equiv 6 \pmod 5$ $\quad$ $1 \equiv -4 \pmod 5$

CHANGING THE SUBJECT OF A FORMULA

In the simple interest formula, $I = \frac{PRT}{100}$, I is the subject of the formula. We can change the subject of the formula to either P, R or T.
For example, to change the subject of the formula to P, divide both sides of the formula by P.

$$\frac{I}{P} = \frac{PRT}{100P} \quad \rightarrow \quad \frac{P}{I} = \frac{100}{RT} \quad \rightarrow \quad P = \frac{100I}{RT}$$

Example:
The formula

$$C = \frac{5(F - 32)}{9}$$

is used to convert between °Fahrenheit and °Celsius.
To rearrange to make F the subject, remove the fraction and multiply both sides by 9

$$9C = 5(F - 32)$$

Expand the brackets and simplify

$$5F = 9C + 160$$

Divide both sides by 5

$$F = \frac{9C + 160}{5}$$

Yet Another Example:
The formula $v^2 = u^2 + 2as$ is used to calculate velocity.
To rearrange to make u the subject, subtract $2as$ from both sides.

$$v^2 - 2as = u^2$$

Take the square root of both sides

$$u = \sqrt{v^2 - 2as}$$

DISTANCE FORMULA

Given the speed and the length of time of a travel, the distance can be found by using the formula:

$$\text{Distance} = \text{speed} \times \text{time}$$

Example:
A Ferrari is driven at its top speed of 349 km/h for 2.5 hours. How far did the car travel?

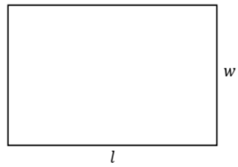
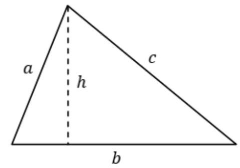
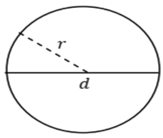
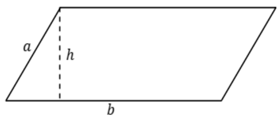
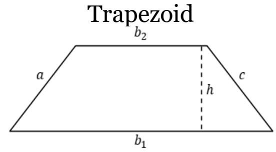
$$\text{Distance} = 349 \times 2.5 = 872.5\ km$$

FUNCTIONS & RELATIONS

A **relation** is a set of ordered pairs between two sets, called the domain and codomain. A **function** is a relation where every element in the domain corresponds to exactly one element in the codomain.
A relation may have multiple outputs for a single input, while a function has exactly one output for every input. All functions are relations, but not all relations are functions. To determine if a relation is a function, check if every element in the domain corresponds to exactly one element in the codomain. If yes, it's a function; if not, it's not a function.
A relation between the sets $\{x, y, z\}$ and $\{1, 2, 3\}$ is: $\{(x, 1), (y, 2), (z, 3), (x, 3)\}$. This relation is not a function because the element 'x' in the domain corresponds to multiple elements (1 and 3) in the codomain.
A function between the sets $\{x, y, z\}$ and $\{1, 2, 3\}$ is: $\{(x, 1), (y, 2), (z, 3)\}$. This is a function because each element in the domain corresponds to exactly one element in the codomain.

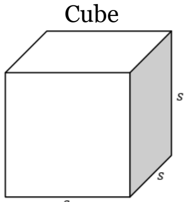
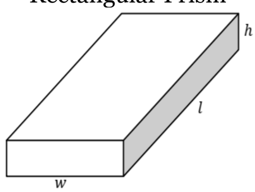
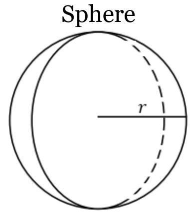
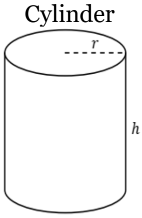
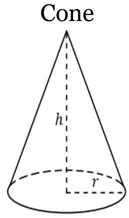
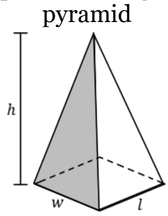
Operations on functions	Given $f(x) = 2x + 3$ and $g(x) = x^2 - 4$
Sum: $(f + g)x = f(x) + g(x)$	$= (2x + 3) + (x^2 - 4) = x^2 + 2x - 1$
Difference: $(f - g)x = f(x) - g(x)$	$= (2x + 3) - (x^2 - 4) = -x^2 + 2x + 7$
Product: $(f \cdot g)x = f(x) \cdot g(x)$	$= (2x + 3)(x^2 - 4) = 2x^3 + 3x^2 - 8x - 12$
Quotient: $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$	$= \frac{2x + 3}{x^2 - 4} \qquad x \neq \pm 4$
Product: $(f \circ g)x = f(g(x))$	$= f(x^2 - 4) = 2(x^2 - 4) + 3 = 2x^2 - 5$

PERIMETER & AREA

Square  $P = 4s$ $A = s^2$	Rectangle  $P = 2(l + w)$ $A = lw$	Triangle  $P = a + b + c$ $A = bh/2$
Circle  $P = C = 2\pi r$ $A = \pi r^2$	Parallelogram  $P = 2(a + b)$ $A = bh$	Trapezoid  $P = a + b_1 + c + b_2$ $A = (b_1 + b_2)h/2$

P = perimeter, *A* = area, *C* = circumference

VOLUME & SURFACE AREA

Cube  $V = s^3$ $S = 6s^2$	Rectangular Prism  $V = lwh$ $S = 2(lh + wh + wl)$	Sphere  $V = 4\pi r^3/3$ $S = 4\pi r^2$
Cylinder  $V = \pi r^2 h$ $S = 2\pi r(h + r)$	Cone  $V = \pi r^2 h/3$ $S = \pi r\sqrt{r^2 + h^2}$	Square/Rectangular pyramid  $V = lwh/3$

V = volume, *S* = surface area

POLYGONS

A polygon is a closed plane figure made up of several line segments that are joined together. Examples include triangles, quadrilaterals, pentagons, hexagons and so on.

	Formulas
The sum of the interior angles	$180°(n - 2)$
The number of diagonals in a polygon	$\frac{n(n-3)}{2}$
The number of triangles in a polygon	$n - 2$
Exterior angle	$\frac{360°}{n}$
Interior angle	$\frac{180°(n-2)}{n}$

Area of a regular polygon	$\frac{ns^2}{4} \times \frac{1}{\tan\left(\frac{180°}{n}\right)}$

$n =$ number of sides and $s =$ side length

Example:

For an octagon with sides equal to 0.765:

$n = 8, s = 0.765$

The sum of the interior angles

$= 180°(8-2) = 1080$

The number of diagonals

$= \frac{8(8-3)}{2} = 20$

The number of triangles $= 8 - 2 = 6$

Exterior angle $= (360°)/8 = 45°$

Interior angle

$= \frac{180°(8-2)}{8} = 135°$

Area

$= \frac{8(0.765)^2}{4} \times \frac{1}{\tan\left(\frac{180°}{8}\right)} \cong 2.83$

VARIATION

Direct or proportional variation: one variable is a constant multiple of another.

For example, the value of y varies directly with x, *or* y is directly proportional to x.

$$y = kx \qquad \text{or} \qquad \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

Inverse or Indirect Variation: when one of the variables increases, the other one decreases.

For example, the value of y varies inversely with x, y is inversely proportional to x, *or* y is indirectly proportional to x

$$y = \frac{k}{x} \quad \text{or} \quad xy = k$$

Joint variation: Like direct variation, but involves more than one variable.

For example, y varies jointly with x and the square of z.

$$y = kxz^2$$

Partial Variation*:* The value of one variable is the sum of two or more quantities each of which is determined by a variation. In some cases, one of those quantities may be constant.

For example, y is partly constant and partly varies directly with x.

$$y = k_1x + k_2$$

Combined Variation: One quantity varies with more than one other variable. This may involve a combination of direct variation or joint variation, and indirect variation.

For example, y varies jointly as x and w and inversely as the square of z.

$$y = \frac{kxw}{z^2}$$

Example:

The simple interest (I) on an investment is jointly proportional to the time (t) and the principal (P). After 5 years, the interest on a principal of \$2000 is \$800. The equation relating the interest, principal, and time is

$$I = kPt$$

Thus,

$800 = k \times 2000 \times 5 \qquad \rightarrow \qquad k = 0.1$

So, the equation relating interest, principal, and time is

$$I = 0.08Pt$$

To find the interest after 10 years

$$I = 0.08 \times 2000 \times 10 = \$1600$$

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